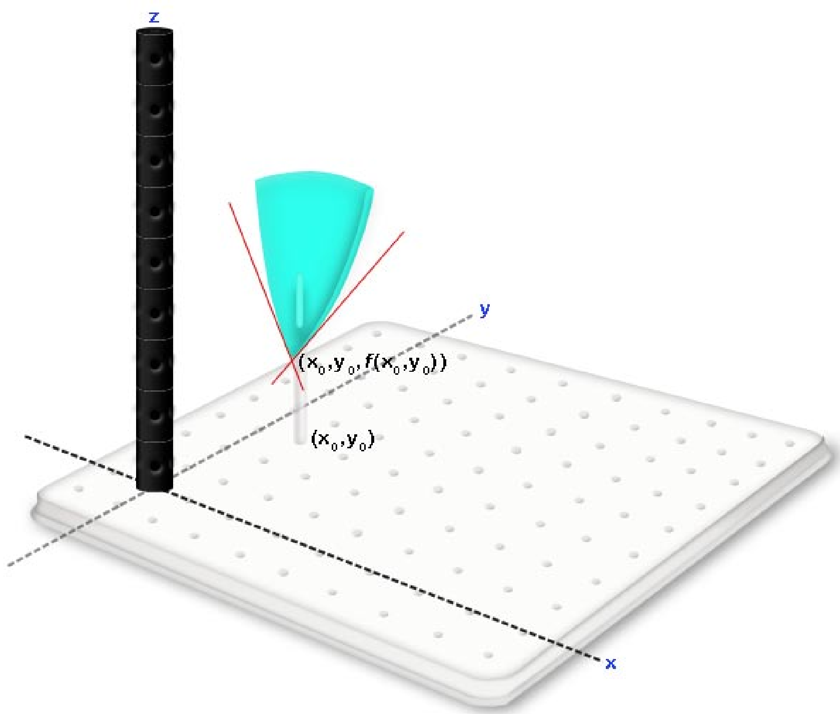
CHAPTER 5:

PARTIAL DERIVATIVES

5.1 DEFINITION OF PARTIAL DERIVATES

GEOMETRIC DEFINITION OF PARTIAL DERIVATES

In our discussion of planes we discussed the slope of the planes in various directions and determined that the slope of a plane may vary depending on the direction in which the slope is taken. Similarly, surfaces $z = f(x,y)$ have different tangent lines in different directions at a given point (x_0, y_0) and the slopes of different tangent lines may vary.

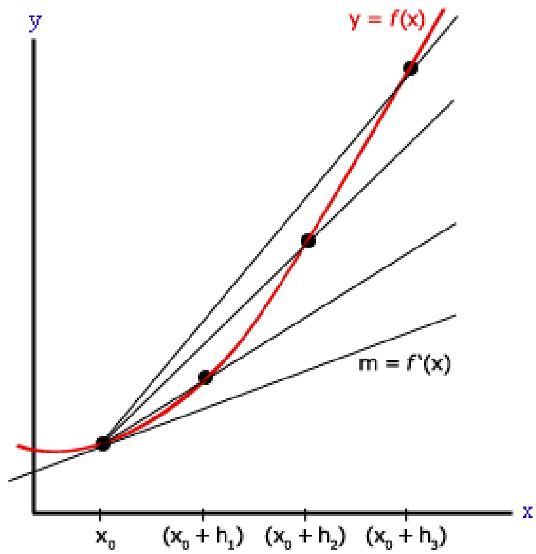


Correspondingly, before we may speak of the derivate or the slope of the tangent line of $z = f(x,y)$, we must first determine the direction of the tangent line we wish to use. With this idea in mind, the definition of partial derivatives in x and y are as follows:

- $\frac{\partial f(a,b)}{\partial x}$ is the slope of the tangent line at the point $(a, b, f(a,b))$ in the x direction.
- $\frac{\partial f(a,b)}{\partial y}$ is the slope of the tangent line at the point $(a, b, f(a,b))$ in the y direction.

REVIEW - ALGEBRAIC DEFINITION OF DERIVATIVES IN TWO DIMENSIONS

In two dimensions, we can obtain the slope of the tangent line of the curve $y = f(x)$ at the point $x = x_0$ by observing a series of secant lines to the curve $y = f(x)$ that begin at the point $(x_0, f(x_0))$. The slope of the line that contains the points $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ is $m = \dfrac{f(x_0 + h) - f(x_0)}{h}$. As the distance between the two points $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ diminishes, the secant line comes to resemble the tangent line.

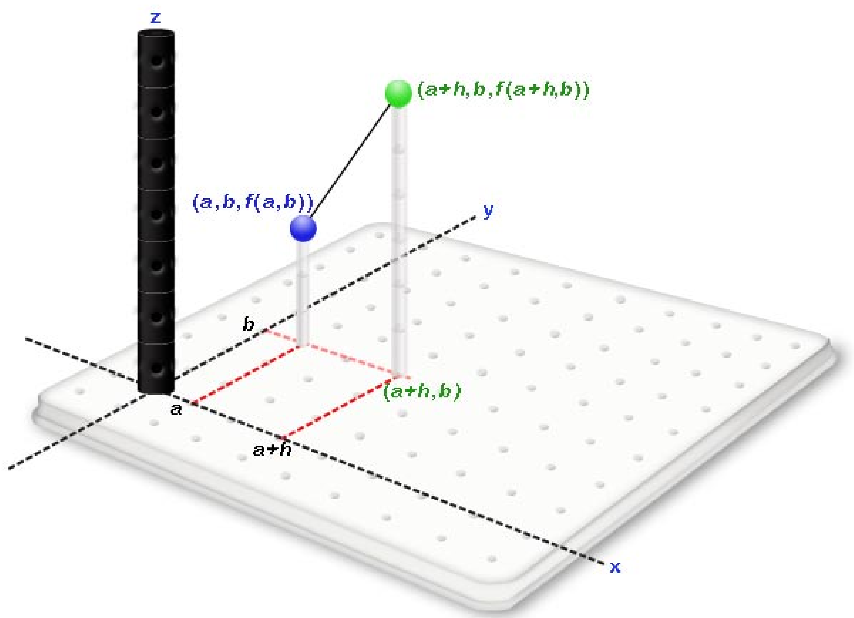


If we continue this process until the two points are indefinitely close together, we can obtain the slope of the tangent line. As the two points become close together, $h \to 0$, while the formula for the slope stays the same. Correspondingly, the algebraic expression for the slope of the tangent line is:

$$f'(x_0) = Lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

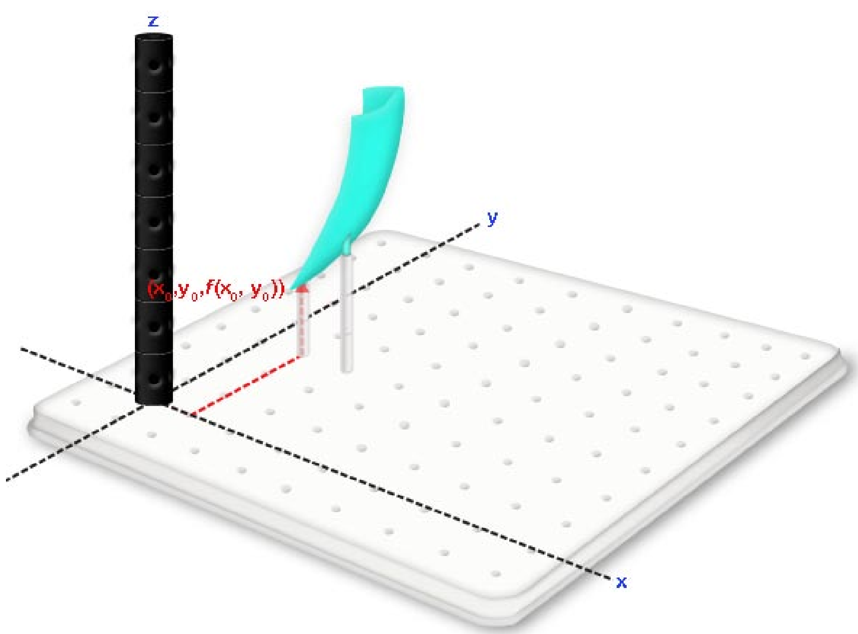
ALGEBRAIC DEFINITION OF THE PARTIAL DERIVATIVE WITH RESPECT TO X

With functions of one variable, we used various secant lines to arrive at the algebraic formula for a derivative. We can follow the same procedure to arrive at the formula for $\dfrac{\partial f(x, y)}{\partial x}$ using only secant lines in the x direction. If our initial point is $(a, b, f(a,b))$, then any point of a secant line in the x direction will have the form $(a + h, b, f(a + h, b))$.

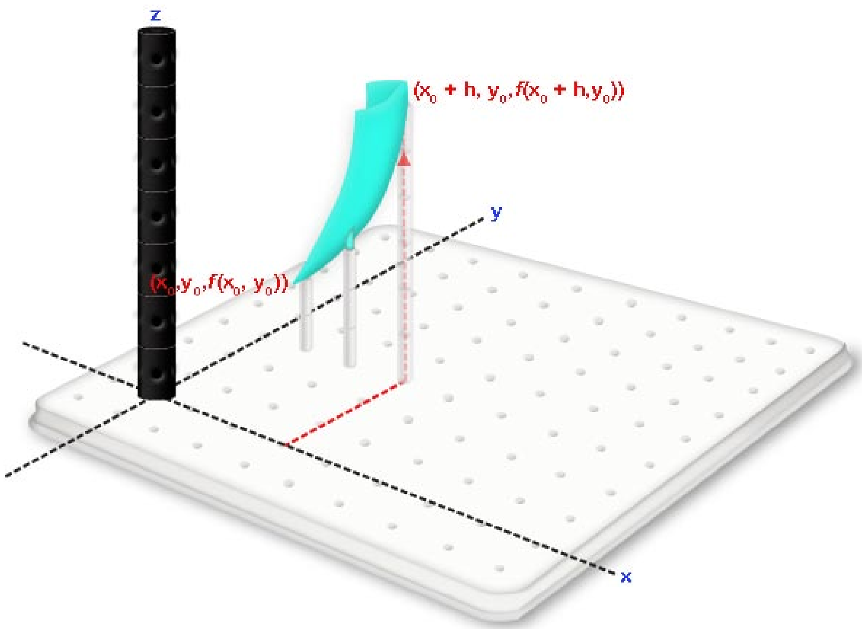


Using such secant lines, we can perform the following steps to obtain the formula $\frac{\partial f(x,y)}{\partial x}$:

1. Place the point $(x_0, y_0, f(x_0, y_0))$ in 3-space.

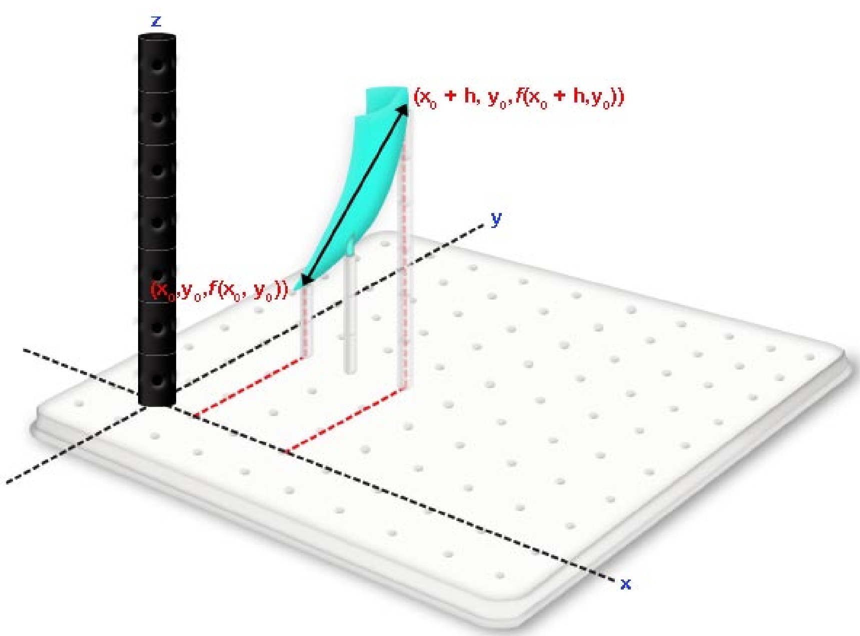


2. Place another point $(x_0 + h, y_0, f(x_0 + h, y_0))$ in the positive x direction from the initial point in 3-space.



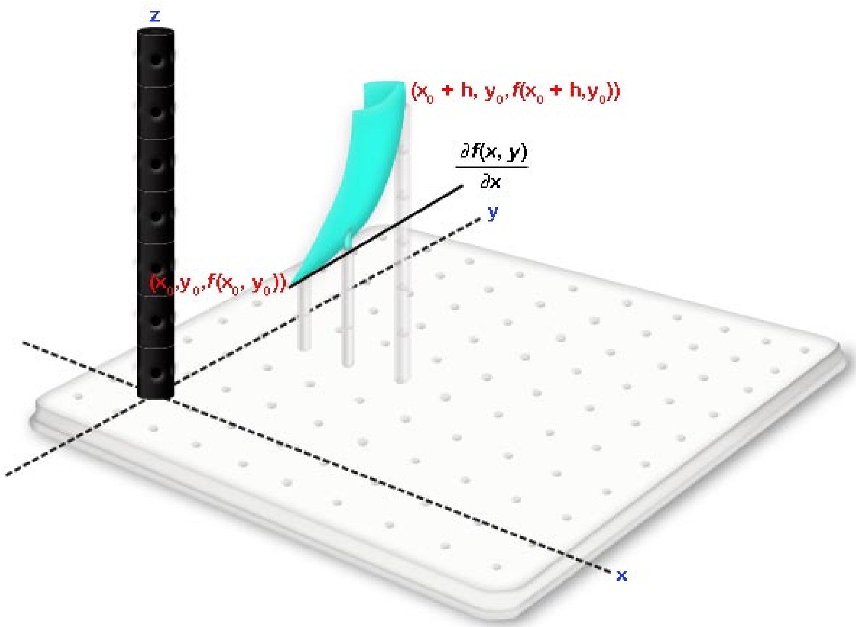
3. Obtain the slope of the secant line between these points.

$$slope = \frac{\Delta z}{\Delta x} = \frac{f(x+h,y) - f(x,y)}{h}$$



4. To obtain the slope of the tangent line in the *x* direction, we can take the limit as *h* goes to zero.

$$slope = \frac{rise}{run} = \frac{\Delta z}{\Delta x} = Lim_{x \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

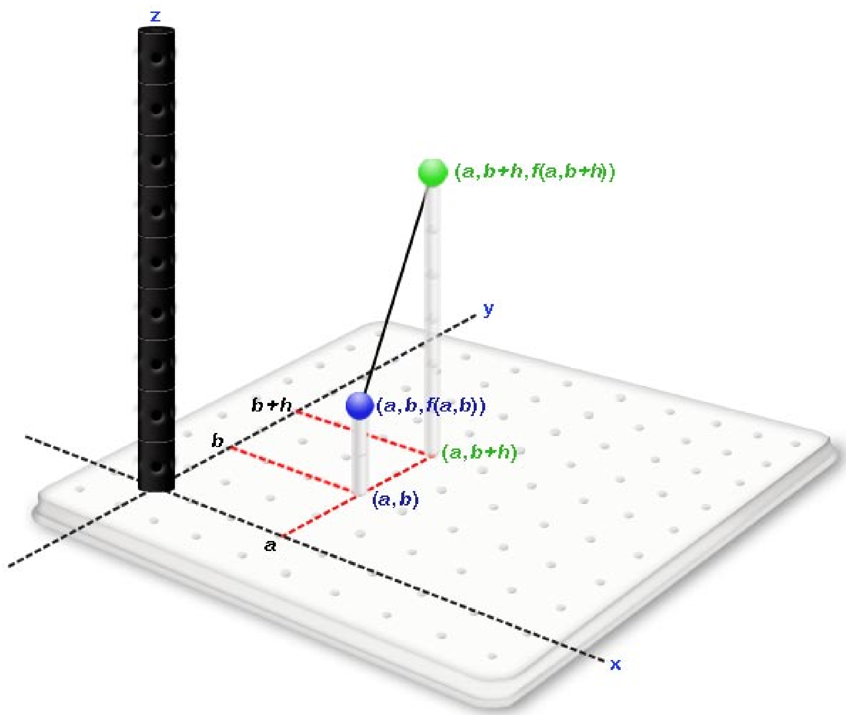


Correspondingly, we can conclude that $\frac{\partial f(x, y)}{\partial x}$ or the slope of the tangent line of $z = f(x, y)$ at the point $(x, y, f(x, y))$ in the x direction is:

$$\frac{\partial f(x, y)}{\partial x} = Lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

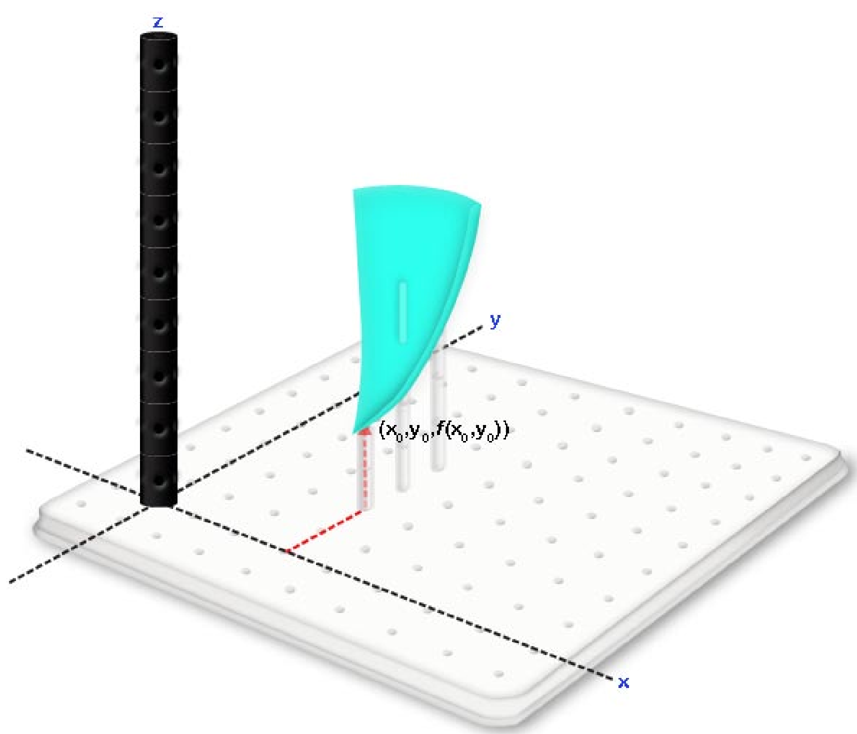
ALGEBRAIC DEFINITION OF PARTIAL DERIVATIVE WITH RESPECT TO Y

To obtain $\frac{\partial f(x, y)}{\partial y}$ or the slope of the tangent line in the y direction, we can use the same procedure with secant lines that we used for partial derivatives with respect to x. Given an initial point is $(a, b, f(a, b))$, any point on the surface $z = f(x,y)$ in the y direction will have the form $(a, b + h, f(a, b + h))$. With these we can construct a secant line in the y direction.

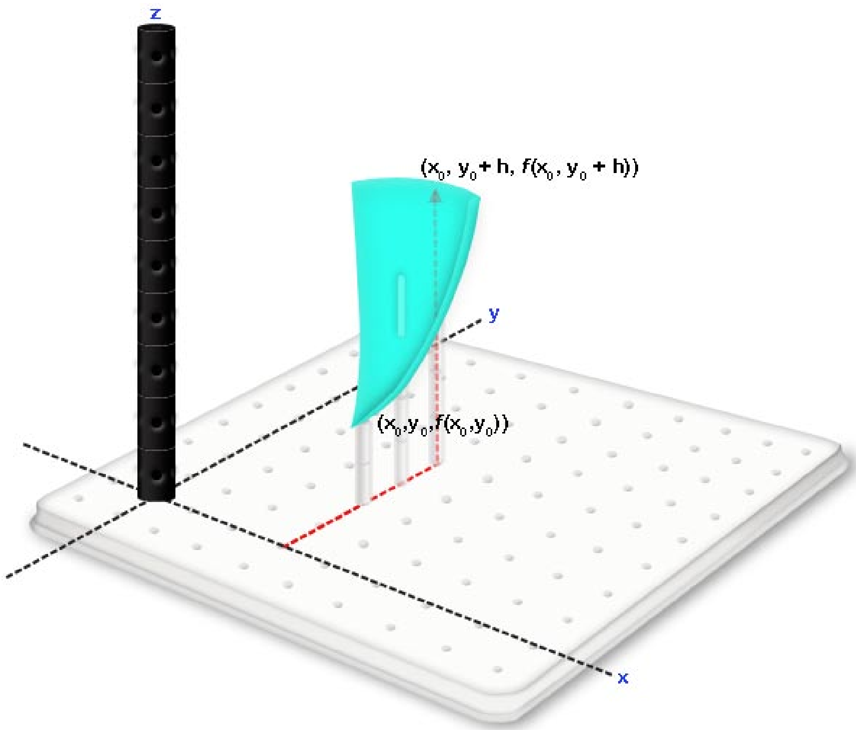


Using secant lines, we can follow the following steps to obtain the definition for $\frac{\partial f(x,y)}{\partial y}$:

1. Place the point $(x_0, y_0, f(x_0, y_0))$ in 3-space.

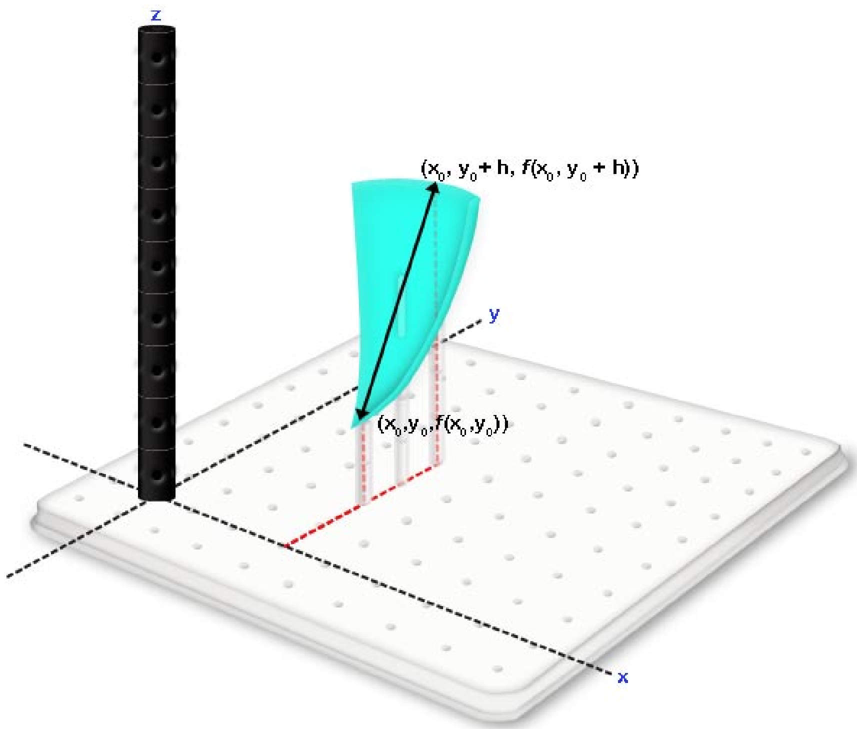


2. Place another point (x_0, $y_0 + h$, $f(x_0, y_0 + h)$), in the positive y direction from the initial point in three space.



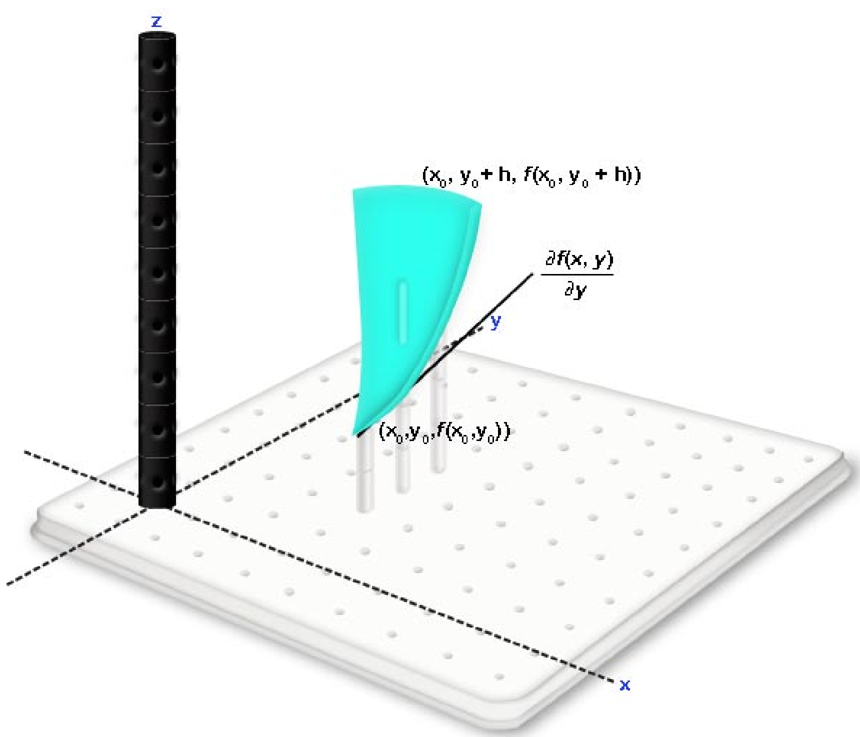
3. Obtain the slope of the secant line between these points.

$$slope = \frac{rise}{run} = \frac{\Delta z}{\Delta y} = \frac{f(x, h+y) - f(x, y)}{h}$$



4. To obtain the slope of the tangent line in the *y* direction, we can take the limit as *h* goes to zero.

$$slope = \frac{rise}{run} = \frac{\Delta z}{\Delta y} = Lim_{h \to 0} \frac{f(x,h+y) - f(x,y)}{h}$$

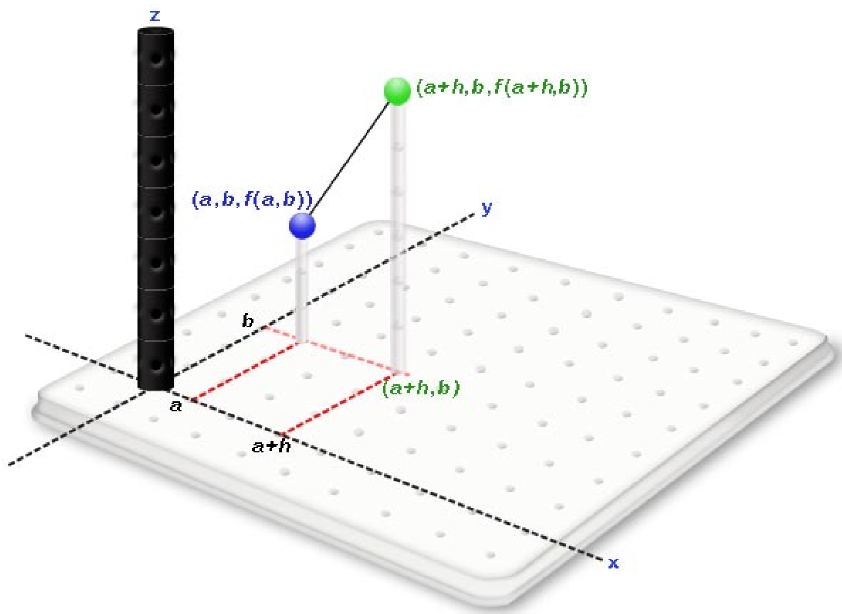


Correspondingly, we can conclude that $\frac{\partial f(x,y)}{\partial y}$ or the slope of the tangent line of $z = f(x,y)$ at the point $(x, y, f(x,y))$ in the y direction is

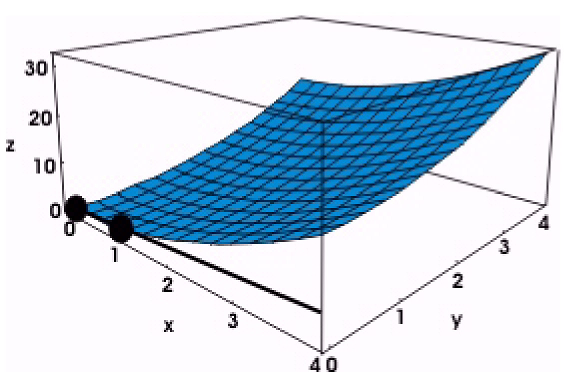
$$\frac{\partial f(x,y)}{\partial y} = Lim_{h\to 0}\frac{f(x,h+y)-f(x,y)}{h}$$

5.2 APPROXIMATING PARTIAL DERIVATIVES

In the previous section, we first established that secant lines to a surface $z = f(x,y)$ at $(a, b, f(a,b))$ in the x direction will pass through the points $(a, b, f(a,b))$ and $(a + h, b, f(a + h, b))$



We then established that as the two points used to obtain the secant line become close together, the slope of the secant line approaches the slope of the tangent line to the surface $z = f(x,y)$ at $(a, b, f(a,b))$ in the x direction.



The guiding principles for approximating $\frac{\partial f(x,y)}{\partial x}$ are

a. secant lines in the *x* direction provide approximations, and

b. the shorter the between the points used to generate the secant line, the better approximation.

Similarly, our guiding principles for approximating $\frac{\partial f(x,y)}{\partial y}$ are

a. secant lines in the *y* direction provide approximations, and

b. the shorter the between the points used to generate the secant line, the better approximation.

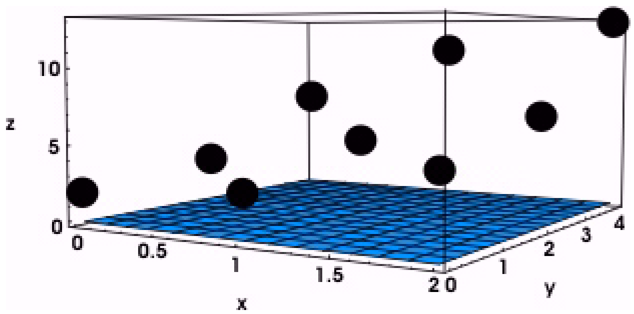
When approximating partial derivatives with points, tables, or contours, these are the guiding principles.

APPROXIMATING PARTIAL DERIVATIVES WITH TABLES

Example Exercise 5.2.1: Given the following table, find **(i)** $\frac{\partial f(1,0)}{\partial x}$ and **(ii)** $\frac{\partial f(1,0)}{\partial y}$

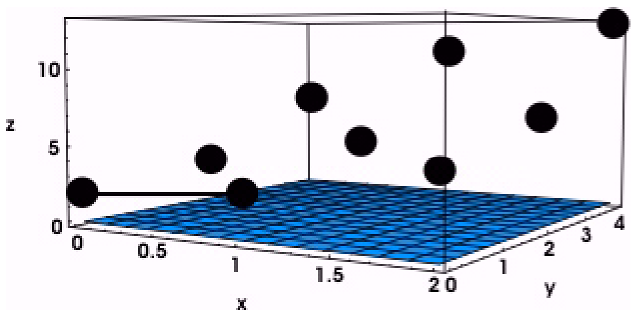
x-y	0	1	2
0	2	3	5
2	3	5	7
4	7	11	13

Solution: The information available in the table is rather limited so it helps to first visualize the information available.



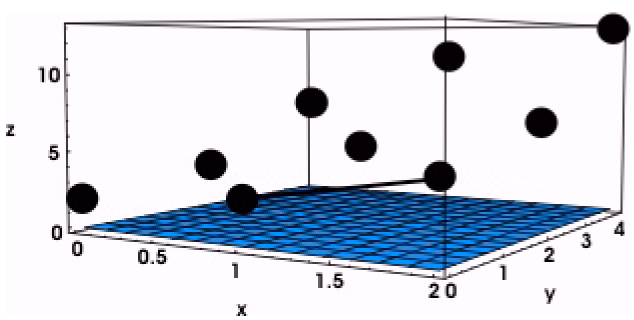
Solution for $\dfrac{\partial f(1,0)}{\partial x}$:

We saw in the preceding section that secant lines can be used to approximate partial derivatives. Due to the limited data available, there are only two secant lines available at the point (1, 0, 3) in the *x* direction. The first of these two lines passes between the points (0, 0, 2) and (1, 0, 3).



Observing that between these two points, $\Delta x = \mathbf{1}$ and $\Delta z = \mathbf{1}$ we can conclude that using these two points, $\dfrac{\partial f(1,0)}{\partial x} = \mathbf{1}$.

The second of these two lines passes between the points (1, 0, 3) and (2, 0, 5).

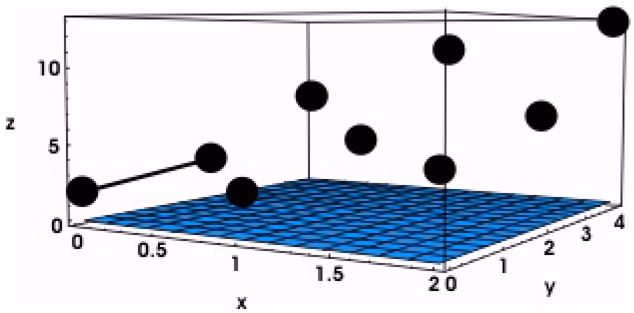


Observing that between these two points, $\Delta x = \mathbf{1}$ and $\Delta z = \mathbf{2}$ we can conclude that using these two points, $\dfrac{\partial f(1,0)}{\partial x} = \mathbf{2}$.

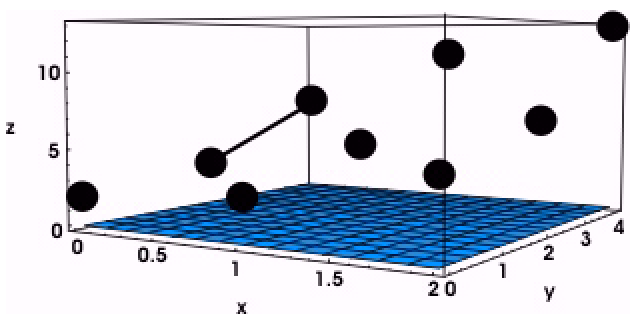
Hence, depending on which secant line we use, we have obtained two approximations for $\frac{\partial f(1,0)}{\partial x}$. The average of these two values $\frac{\partial f(1,0)}{\partial x} = \frac{1+2}{2} = 1.5$ can also be used.

Solution for $\frac{\partial f(1,0)}{\partial y}$:

Similarly, there are only two secant lines available at the point (0, 2, 3) in the *y* direction. The first of these two lines passes between the points (0, 0, 2) and (0, 2, 3).



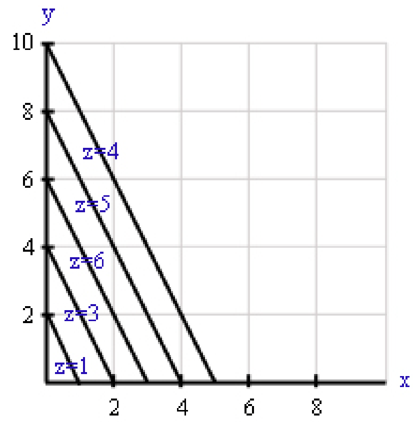
Observing that between these points, $\Delta y = 2$ and $\Delta z = 1$ we can conclude that using these two points, $\frac{\partial f(1,0)}{\partial y} = \frac{1}{2}$.



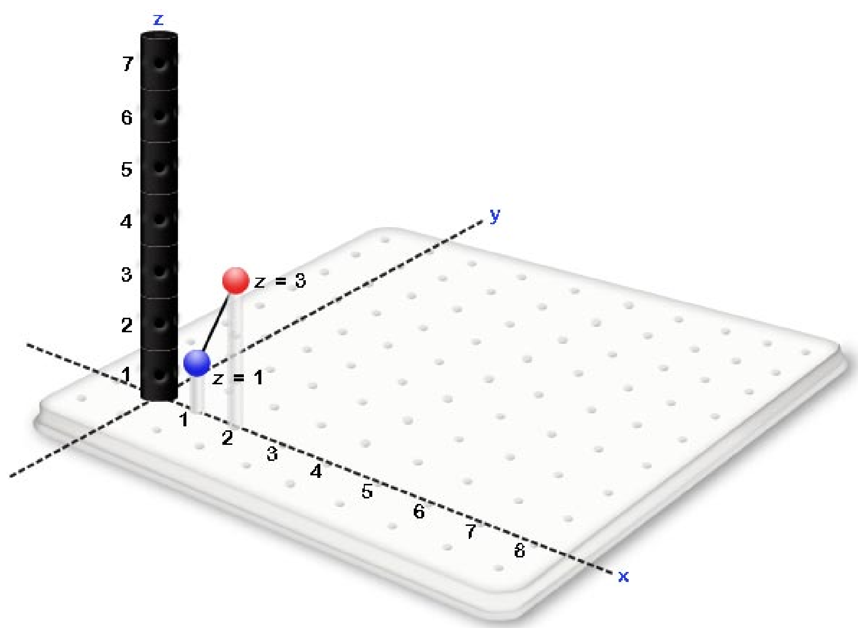
Observing that between these two points, $\Delta y = 2$ and $\Delta z = 4$ we can conclude that using these two points $\frac{\partial f(1,0)}{\partial y}$ = 2. Hence, depending on which secant line we use, we have obtained two approximations for $\frac{\partial f(1,0)}{\partial y}$. The average of these two values $\frac{\partial f(1,0)}{\partial y} = \frac{0.5+2}{2} = 1.25$ can also be used.

APPROXIMATING PARTIAL DERIVATIVES WITH CONTOURS

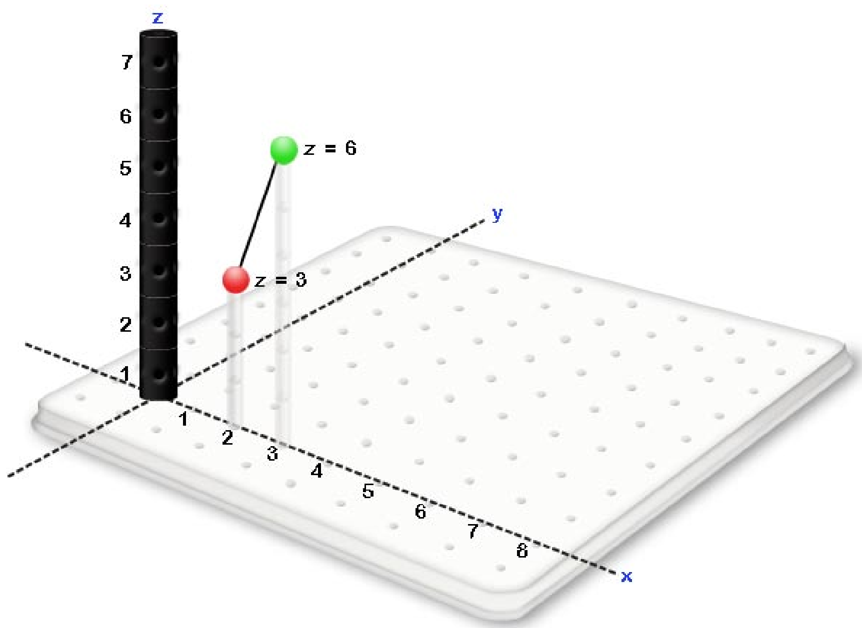
Example Exercise 5.2.2: Given the following contour diagram, find (i) $\frac{\partial f(2,0)}{\partial x}$ and (ii) $\frac{\partial f(2,2)}{\partial y}$



Solution for $\frac{\partial f(2,0)}{\partial x}$ **:** There are only two secant lines available at the point (2, 0, 3) in the *x* direction. The first of these two lines passes between the points (1, 0, 1) and (2, 0, 3).

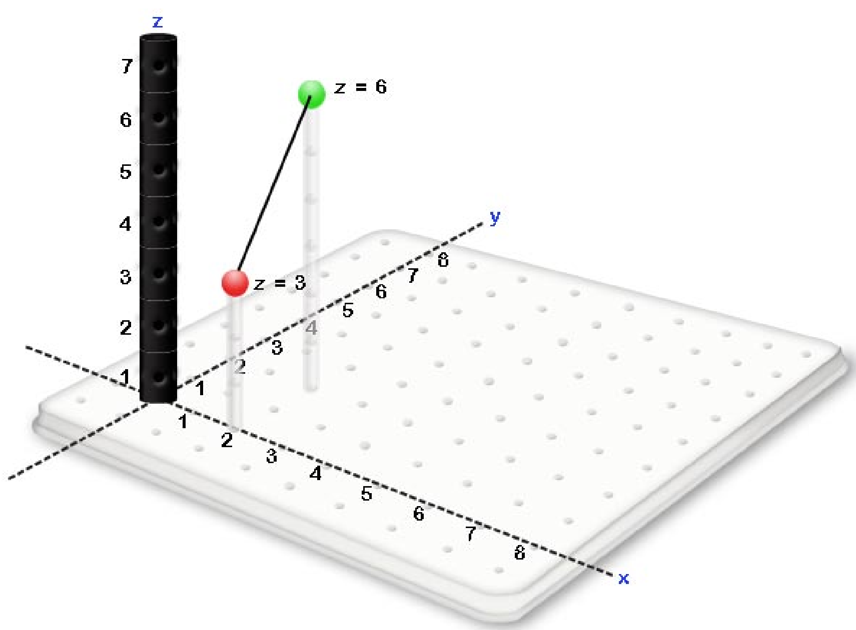


Observing that between these two points, $\Delta x = 1$ and $\Delta z = 2$ we can conclude that using these two points, $\frac{\partial f(2,0)}{\partial x} = \mathbf{2}$. The second of these two secant lines passes between two points (2, 0, 3) and (3, 0, 6).

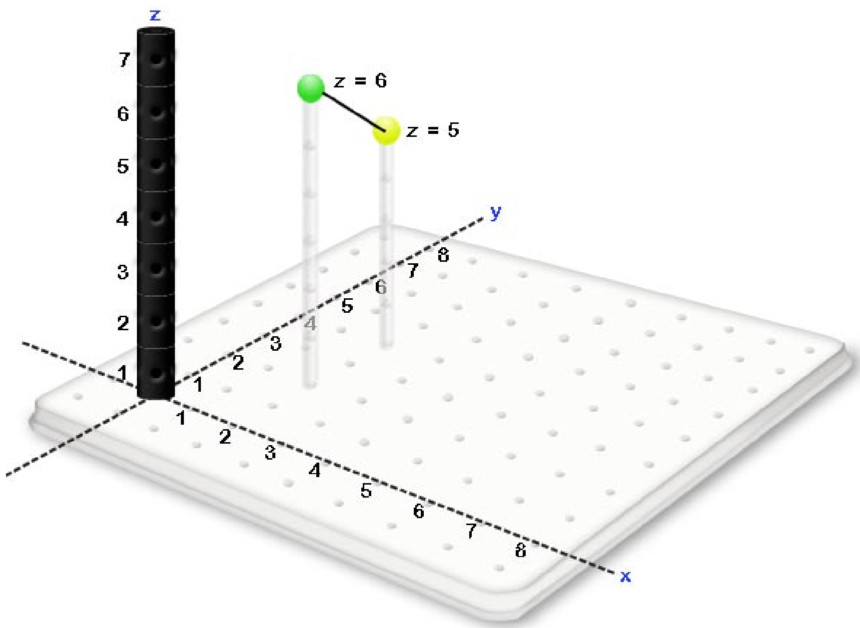


Observing that between two points, $\Delta x = 1$ and $\Delta z = 3$ we can conclude that using these two points, $\frac{\partial f(2,0)}{\partial x} = \mathbf{3}$. Hence, depending on which secant line we use, we have obtained two approximations for $\frac{\partial f(2,0)}{\partial x}$. The average of these two values $\frac{\partial f(2,0)}{\partial x} = \frac{2+\mathbf{3}}{\mathbf{2}} = 2.5$ can also be used.

Solution for $\frac{\partial f(2,2)}{\partial y}$ **:** Similarly, there are only two secant lines available at the point (2, 2, 6) in the *y* direction. The first of these two lines passes between the points (2, 0, 3) and (2, 2, 6).



Observing that between these two points, $\Delta y = 2$ and $\Delta z = 3$ we can conclude that using these two points, $\frac{\partial f(2,2)}{\partial y} = \frac{3}{2}$. The second of these two secant lines passes between the points (2, 2, 6) and (2, 4, 5).



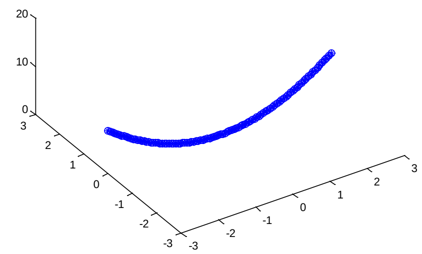
Observing that between these two points, $\Delta y = 2$ and $\Delta z = 1$ we can conclude that using these two points, $\frac{\partial f(2,2)}{\partial y} = \frac{-1}{2}$. Hence depending on which secant line, we have obtained two approximations for $\frac{\partial f(2,2)}{\partial y}$. The average of these two values $\frac{\partial f(2,2)}{\partial y} = \frac{1.5 + (-0.5)}{2} = \frac{1}{2}$ can also be used.

5.3 FORMULAS FOR PARTIAL DERIVATIVES

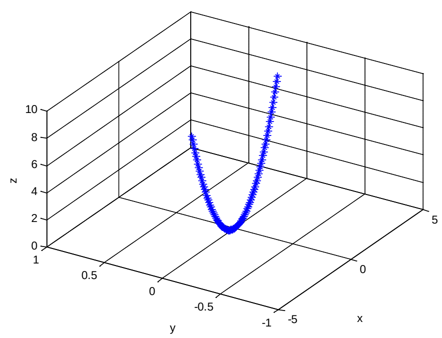
PARTIAL DERIVATIVE WITH RESPECT TO X

Example Exercise 5.3.1: Find the partial derivative with respect to x of $f(x,y) = x^2 + y^2$ at the point (0, 0, 0).

Solution: We are interested in the slope of the tangent line in the x direction so the only section of the surface that is of interest is the curve at the point (0, 0, 0) associated with movement in the x direction. See the following figure:



If we are moving in the x direction from the point (0, 0, 0) then the value of y remains constant at $y = 0$. Hence, the only part of the surface we need for this problem is the cross section $y = 0$ where $z = x^2 + 0^2$.



The slope of the tangent line to $z = x^2 + 0^2$ in the x direction can be obtained by the derivative from first year calculus $\frac{\partial z}{\partial x} = \mathbf{2x}$. When $x = 0$, $\frac{\partial f(0,0)}{\partial x} = 2 * 0 = \mathbf{0}$. This is consistent with the slope that can be observed geometrically.

Generalization: Given a function f represented by a formula $z = f(x,y)$, we can calculate the partial derivative with respect to x at a point (a, b) by

1. Going to the point $(a, b, f(a,b))$ on the surface.
2. Recognizing that for this problem we are interested in the rate of change associated with movement in the x direction.
3. When we move in the x direction, y remains constant at the value $y = b$. Hence the curve $z = f(x,b)$ is all we need for this problem

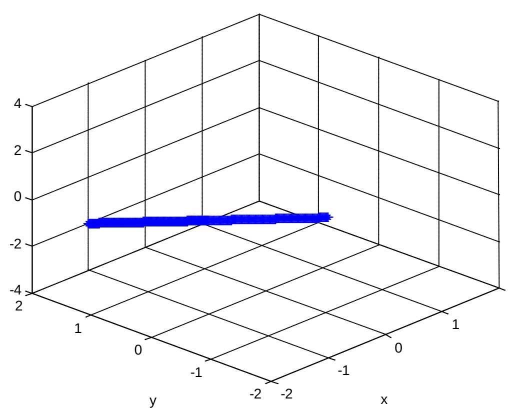
z = f(x , b)

4. The derivative of the tangent line in the x direction of the curve $z = f(x,b)$ when $x = a$ can be obtained with a straightforward derivative from first year calculus.

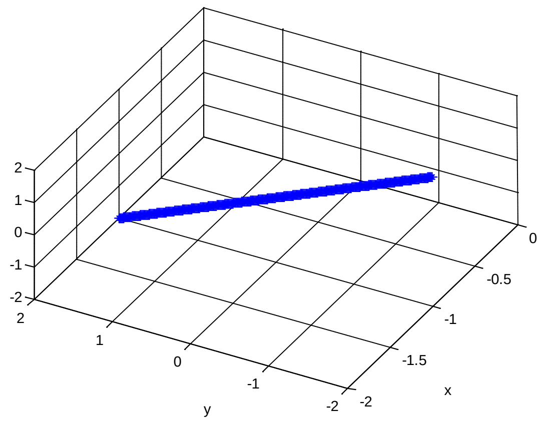
PARTIAL DERIVATIVE WITH RESPECT TO Y

Example Exercise 5.3.2: Find the partial derivative with respect to y of $f(x,y) = xy$ at the point (-1, 1, -1).

Solution: We are interested in the slope of the tangent line in the x direction so the only section of the surface that is of interest is the curve at the point (-1, 1,-1) associated with movement in the y direction. See the following figure:



If we are moving in the y direction from the point (-1, 1,-1) then the value of x remains constant at $x = -1$. Hence, the only part of the surface we need for this problem is the cross section $x = -1$ where $z = -1y$.



The slope of the tangent line to z = -1y in the y direction can be obtained by the derivative from first year calculus $\frac{\partial z}{\partial y} = -1$. When y=1, $\frac{\partial f(-1,1)}{\partial y} = -1$. This is consistent with the slope that can be observed geometrically.

Generalization

Given a function f represented by a formula $z = f(x,y)$, we can calculate the partial derivative with respect to y at a point (a, b) by

1. Going to the point (a, b, $f(a,b)$) on the surface.
2. Recognizing that for this problem we are interested in the rate of change associated with movement in the y direction.
3. When we move in the y direction, x remains constant at the value $x = a$. Hence the curve $z = f(a,y)$ is all we need for this problem

z = f(a, y)

4. The derivative of the tangent line in the y direction of the curve $z = f(a,y)$ when $y = b$ can be obtained with a straightforward derivative from first year calculus.